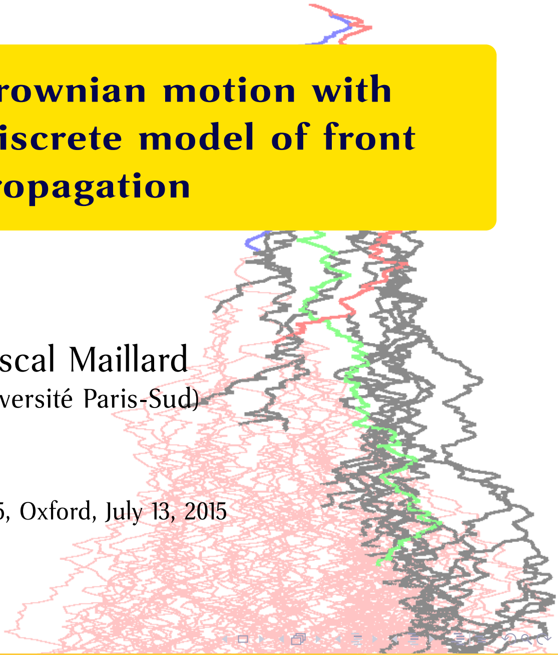
Branching Brownian motion with selection: a discrete model of front propagation



Pascal Maillard
(Université Paris-Sud)

SPA 2015, Oxford, July 13, 2015

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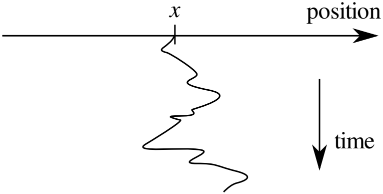
Definition

- A particle performs standard Brownian motion started at a point $x \in \mathbb{R}$.



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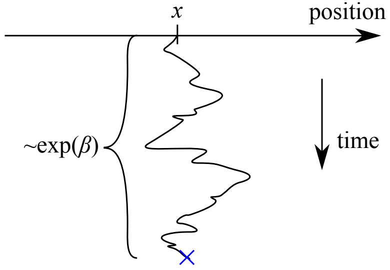
- With rate β, it branches, i.e. it dies and spawns L offspring (L being a random variable).



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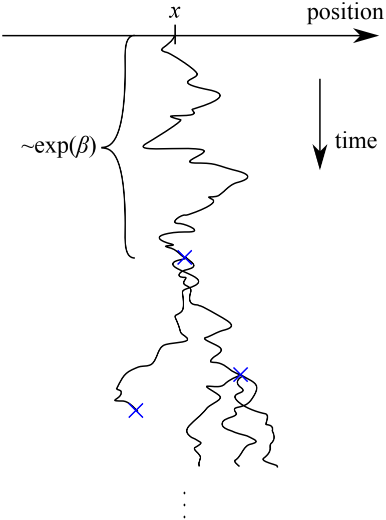

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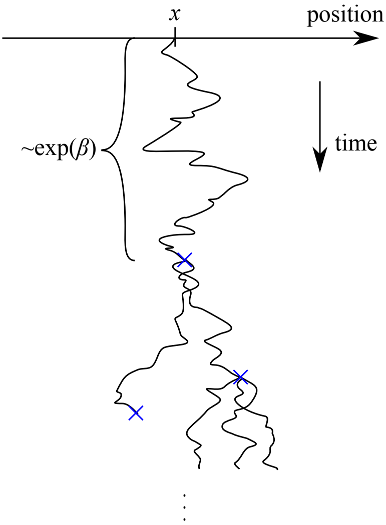
$\longrightarrow$ A **Brownian motion** indexed by a **tree**.



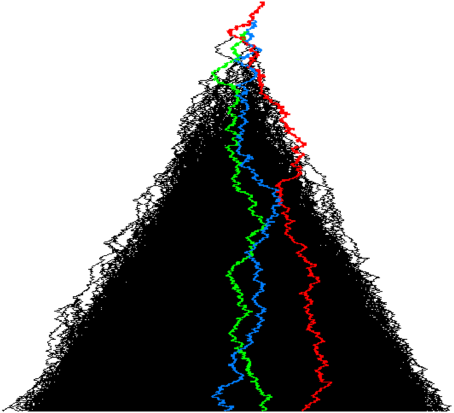
Branching Brownian motion (BBM) (2)

We always suppose
$m := \mathbb{E}[L] - 1 > 0$.

Right-most particle

Let R_t be the position of the right-most particle. Then, as $t \to \infty$, almost surely on the event of survival,

$$\frac{R_t}{t} \to \sqrt{2\beta m}.$$



Picture by Éric Brunet

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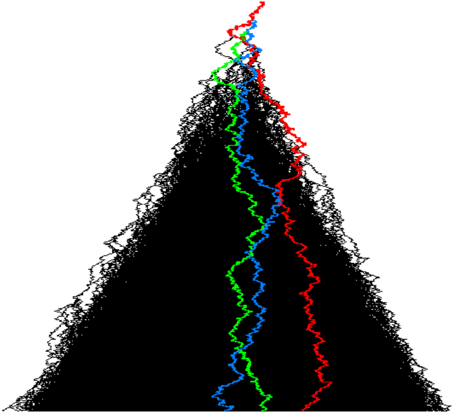
Convention

We will henceforth set $\beta = 1/(2m)$.



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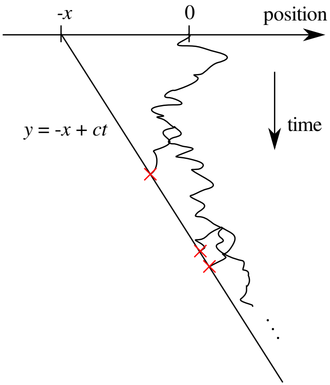
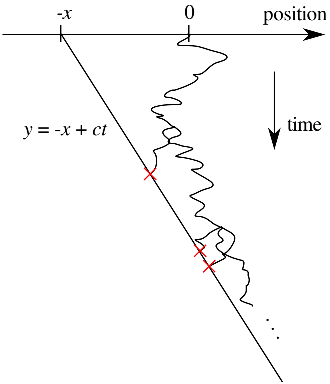
Selection



Two models of BBM with **selection**:

Selection



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1. **BBM with absorption**: Let $f(t)$ be a continuous function (the **barrier**). Kill an individual as soon as its position is less than $f(t)$.

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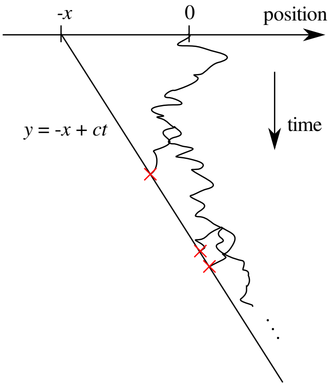
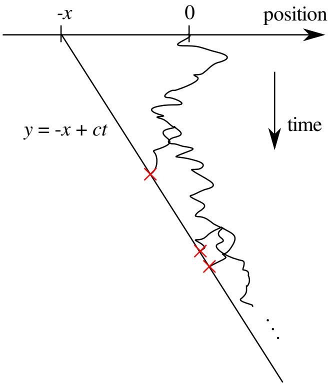


Two models of BBM with **selection**:

1. **BBM with absorption**: Let $f(t)$ be a continuous function (the **barrier**). Kill an individual as soon as its position is less than $f(t)$.
2. **BBM with constant population size (N-BBM)**: Fix $N \in \mathbb{N}$. As soon as the number of individuals exceeds N, only keep the N right-most individuals and kill the others.

Branching Brownian motion with absorption

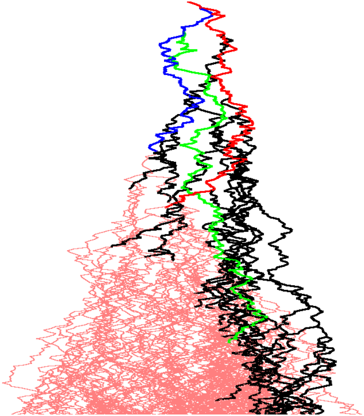


We take $f(t) = -x + ct$ (*linear barrier*).
Vast literature, known results (sample):

- almost sure extinction $\Leftrightarrow c \geq 1$
  ($c = 1$: critical case
  $c > 1$: supercritical case)
- growth rates for $c < 1$.
- asymptotics for extinction probability for $c = 1 - \varepsilon$, ε small

Exact formulae for many quantities of interest.

BBM with constant population size



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Recall: Fix $N \in \mathbb{N}$. As soon as the number of individuals exceeds N, only keep the N right-most individuals and kill the others.

Much less tractable than BBM with absorption:

- strong interaction between particles
- no exact formulae

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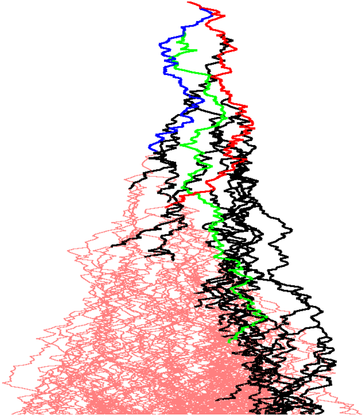

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Nevertheless: A fairly detailed heuristic picture developed in the physics literature over the course of ten years:

Brunet and Derrida (1997-2004)

with Mueller and Munier (2006-2007)

Heuristic picture of N-BBM BDMM '06

- Meta-stable state: cloud of particles moving at speed $v_N^{\text{det}} = \sqrt{1 - \pi^2/\log^2 N}$, empirical measure seen from the left-most particle approximately proportional to $\sin(\pi x/\log N)e^{-x}\mathbf{1}_{(0,\log N)}(x)$, diameter $\approx \log N$.

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Real speed of the system is approximately

$$v_N = \sqrt{1 - \frac{\pi^2}{a_N^2}} = v_N^{\text{det}} + \frac{3\pi^2 \log\log N + o(1)}{\log^3 N},$$

and $O(1/\log^3 N)$ fluctuations.

Rigorous results (1)

- Bérard and Gouéré (2012): Prove the $1/\log^2 N$ correction term to v_N for more general branching random walks (relying on results about BRW absorbed at a linear barrier by Gantert, Hu and Shi (2011)).
- 2×Berestycki and Schweinsberg (2013): Study BBM with absorption at a linear barrier with slope v_N → toy model for N-BBM. They show convergence of the genealogy (as $N \to \infty$) to the Bolthausen–Sznitman coalescent, on time scale $(\log N)^3$.
- Durrett and Remenik (2010): Study empirical measure seen from left-most particle in a certain N-BRW. Show convergence of its evolution to a certain free-boundary convolution equation *(without rescaling in time)*.
- Mueller, Mytnik and Quastel (2010): Prove $O(\log\log N/\log^3 N)$ correction term for noisy FKPP equation.

Rigorous results (2)

$a_N = \log N + 3 \log\log N$, $v_N = \sqrt{1 - \pi^2 / a_N^2}$

$X_i(t)$: position of i-th particle to the right at time t.

Theorem (M. '13+)

Suppose $\mathbb{E}[L^2] < \infty$ and at time 0, there are N particles distributed independently in $(0, a_N)$ according to density proportional to $\sin(\pi x / a_N) e^{-x}$. Then, for every $\alpha \in (0,1)$,

$$\left(X_{\alpha N}(t \log^3 N) - v_N t \log^3 N\right)_{t \geq 0} \overset{fidis}{\Longrightarrow} (L_t)_{t \geq 0}.$$

Here, $(L_t)_{t \geq 0}$ is a Lévy process with $L_0 = x_\alpha$ (explicit), a certain (non-explicit) drift and explicit Lévy measure (the image of $\pi^2 x^{-2} \mathbf{1}_{x>0}\,\mathrm{d}x$ by the map $x \mapsto \log(1+x)$).

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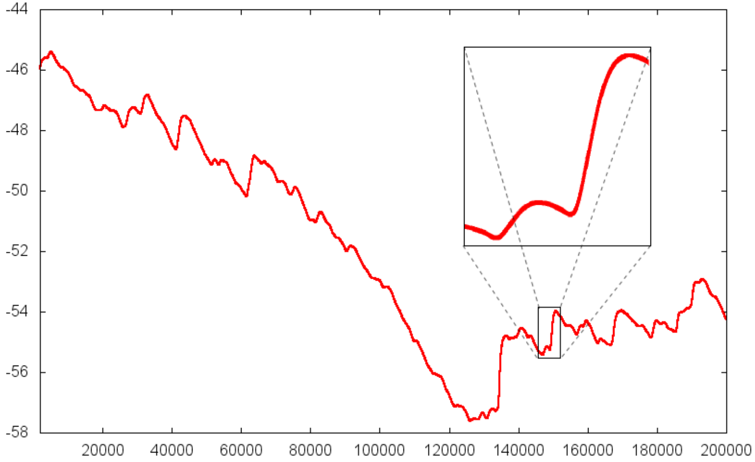
Proof idea: Couple the N-BBM with BBM with a certain (random) absorbing barrier.

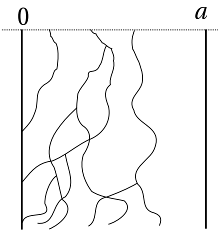
Simulation – Recentered position of barycenter



10^{10} particles

The B-BBM: the approximate model

$a = \log N + 3\log\log N - A$: Position of a second barrier (idea from BBS '10).
Drift: $-\sqrt{1-\pi^2/a^2}$.
Let first N, then A go to ∞.



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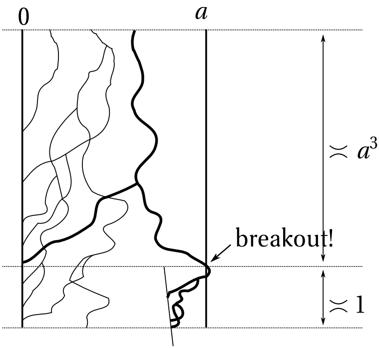
When particle hits a, it will create $\asymp e^{-A} WN$ descendants, where (BBS '10)

$$\mathbb{P}(W > x) \sim x^{-1}, \quad x \to \infty.$$

Breakout when $W > \varepsilon e^A$, ε small.



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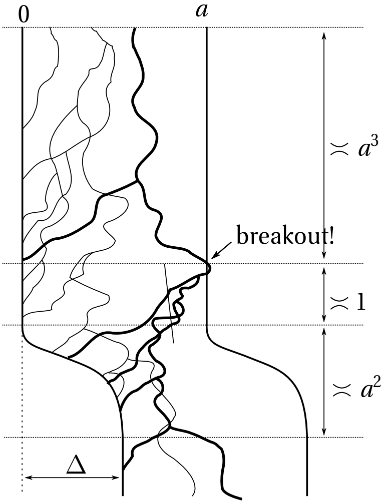
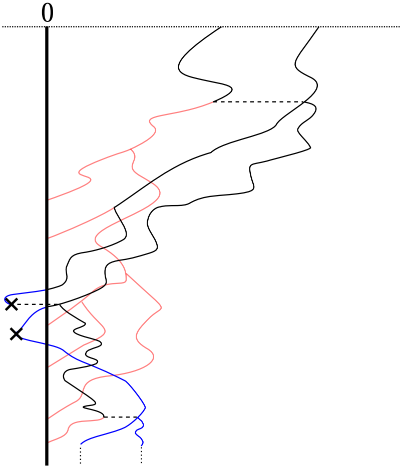
After breakout, move barrier smoothly by random amount $\Delta \approx \log(1 + W)$.



B-BBM $\leftrightarrow$ N-BBM

First idea: couple both processes.

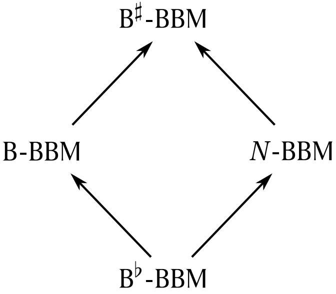
- **black** particles: present in B-BBM **and** N-BBM,
- red particles: present in B-BBM but **not** in N-BBM,
- blue particles: present in N-BBM but **not** in B-BBM.



Problem

Dependencies between particles too difficult to handle.

The solution



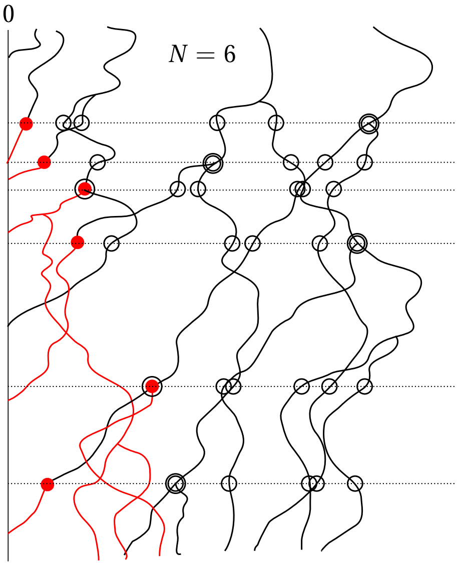
Introduce two auxiliary particle systems: The B$^\flat$-BBM and the B$^\sharp$-BBM (stochastically) bound the N-BBM (and the B-BBM) from below and above (in the sense of stochastic order on the empirical measures).

Bounding the N-BBM from below: The $B^\flat$-BBM

Kill a particle

- whenever it hits 0 **or**
- whenever it has N particles to its right (red particles).

$\Longrightarrow$ more particles are being killed than in N-BBM.



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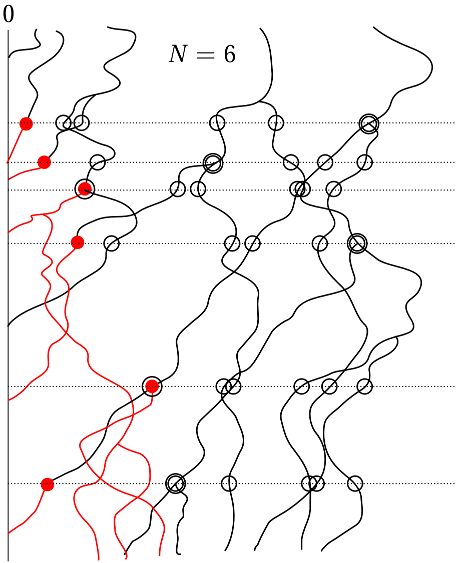


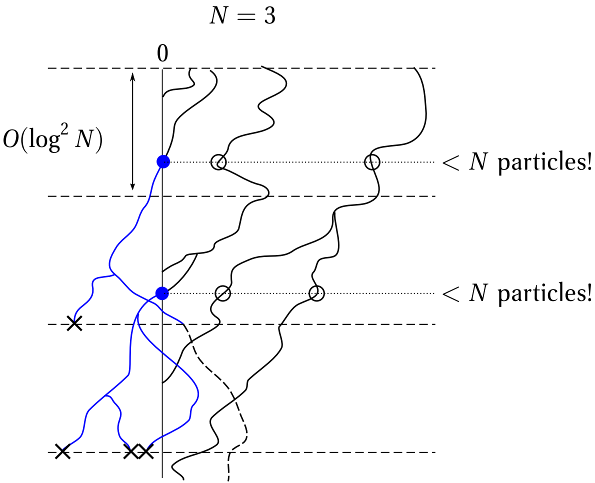
At timescale $\log^3 N$, number of red particles stays negligible.

Bounding the N-BBM from above: The B$^\sharp$-BBM

Kill a particle whenever it (at the same time)

- hits 0 **and**
- has N particles to its right.

A particle survives temporarily (blue particles) if it has less than N particles to its right the moment it hits 0.



N-BBM open problems

- Long-time behavior:
  - exact speed asymptotics
  - empirical measure under equilibrium
  - relaxation time of empirical measure
- Genealogy
  - Show convergence to Bolthausen–Sznitman coalescent (at timescale $\log^3 N$). Proven for BBM with near-critical absorption (BBS '10) and for another N particle model called the *exponential model* (Brunet–Derrida, Comets–Ramirez–Quastel, Cortines)
- Durrett–Remenik free boundary equation
  - convergence to travelling wave

Related works/models

- N. Berestycki, Zhao '14: d-dimensional N-BBM (keep N particles with largest modulus). Show existence of a cloud of particles of width $\log N$ and length $(\log N)^{3/2}$ moving at linear speed in a uniformly chosen direction.
- Mallein '15: BBM (actually, branching random walk), fix $c > 0$. At time t, keep only $N_t = \exp(ct^{1/3})$ right-most particles (then $ct = (\log N)^3$). Position of right-most particle at time t:

$$t - \frac{3\pi^2}{2a^2} t^{1/3} + o(t^{1/3}).$$

Related works/models (2)

Mallein '15: BRW with slightly heavier tails ($\mathbb{E}[e^X] < \infty$ but possibly $\mathbb{E}[X^2 e^X] = \infty$) in the following regime:

- still linear speed of right-most particle
- path of right-most particle "almost" an excursion of an α-stable Lévy process, $\alpha \in (0, 2]$.

Considers N-BRW with these parameters. Shows that for some slowly varying function $L(x)$,

$$1 - v_N \sim \frac{L(\log N)}{(\log N)^\alpha}.$$

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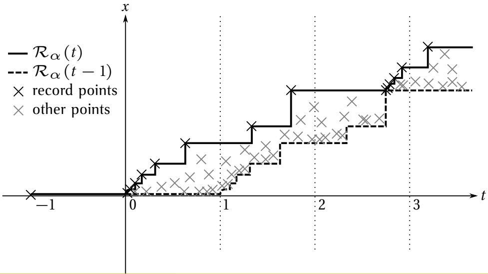
Note: in all of these works, basic tool is coupling with BRW/BBM with absorption at a linear barrier.

Bérard, M. '14: N-BRW with regularly varying tails (e.g. $\mathbb{P}(X > x) \sim x^{-\alpha}$, $\alpha > 0$). Binary branching. Phenomenology much different than N-BBM:

- Typically, most of the N particles are located near the minimum.
- From this position, single particles jump to higher positions and create new "colonies".
- A colony reaches a population size of order N after time $\log_2 N$ (if it survived that long). At this time, it overtakes the whole population.
- For a colony to reach this size, it has to be created at a new record position.



Limiting behavior decribed by a real-valued process $\mathcal{R}_\alpha(t)$ constructed out of a Poisson process. A consequence ($\alpha > 1$):

$$v_N \sim v_{\mathcal{R}_\alpha} (2N \log N)^{1/\alpha} / \log N.$$

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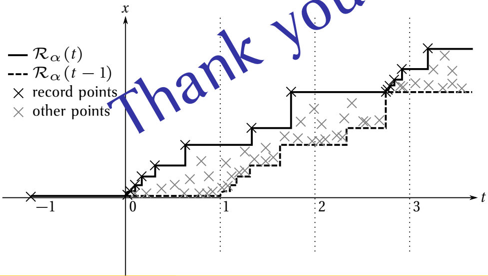
Thank you for your attention!



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BBM $\longleftrightarrow$ FKPP

Let $g : \mathbb{R} \to [0,1]$ be measurable. Define

$$u(t,x) = \mathbb{E}_x\Big[\prod_{u\in\mathcal{N}_t} g(X_u(t))\Big].$$

Then u satisfies the following partial differential equation:

Fisher–Kolmogorov–Petrovskii–Piskunov (FKPP) equation

$$\begin{cases} \partial_t u = \frac{1}{2}\partial_x^2 u + \beta(\mathbb{E}[u^L] - u) \\ u(0,x) = g(x) \quad \text{(initial condition)} \end{cases}$$

The prototype of a parabolic PDE admitting travelling wave solutions.

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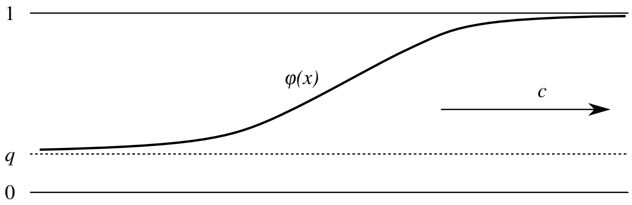
Duality between BBM and FKPP.

Travelling waves



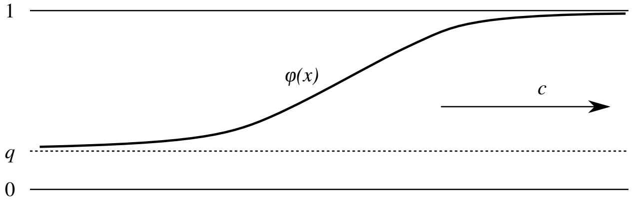
Definition

A *travelling wave* of speed c is a solution of the FKPP equation of the form

$$u(t,x) = \phi(x - ct),$$

where $\phi(x)$ is an increasing function with $\phi(\infty) = 1$ and $\phi(-\infty) = q$, where q solves $\mathbb{E}[q^L] = q$.

Travelling waves



Theorem (KPP '37)

- *Travelling waves exist for every speed $c \geq 1$ and are unique up to translation.*

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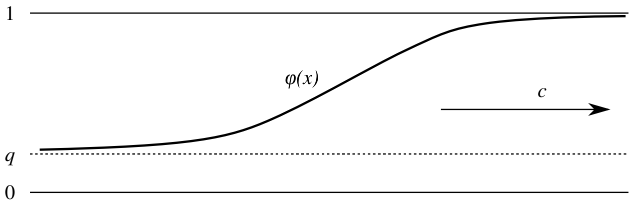

Theorem (KPP '37)

- *Travelling waves exist for every speed $c \geq 1$ and are unique up to translation.*
- *Starting from Heaviside initial data $u(0,x) = 1_{\{x \geq 0\}}$, there exists a centering term $m(t)$, such that*

$$u(t, x + m(t)) \xrightarrow{t \to \infty} \phi_1(x).$$

N-BBM $\longleftrightarrow$ noisy FKPP

Noisy FKPP equation

$$\begin{cases} u(t,x) : \mathbb{R}_+ \times \mathbb{R} \to [0,1] \\ \partial_t u = \partial_x^2 u + u(1-u) + \sqrt{\varepsilon u(1-u)}\,\dot{W} \\ u(0,x) = \mathbf{1}_{(x<0)} \qquad \text{(IC)} \end{cases}$$

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- Admits travelling wave solutions with same phenomenology as N-BBM ($N \simeq \varepsilon^{-1}$) Mueller, Mytnik and Quastel '10